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NATURALLY REDUCTIVE LOCALLY 4-SYMMETRIC SPACES

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ABSTRACT

Naturally reductive Riemannian 3-symmetric spaces have been shown to relate closely to the class of nearly Kahler manifolds. Our purpose here is to introduce a similar study of 4-symmetric spaces by considering a particular F-struture which they all carry. This appears to be the natural analogue of a nearly Kahler structure. The main theorem provides a characterisation of all naturally reductive locally 4-symmetric spaces in terms of the associated F-structure.

1. Introduction

Riemannian k-symmetric spaces form a natural extension to the symmetric spaces of E. Cartan. Basic properties and methods for the classification of these spaces can be found in [7], [12], [13] and [14]. More detailed accounts have been given in [4] and [5] for the cases $k = 3$ and $k = 4$ respectively. Also a full classification for spaces of dimension $n \leqq 5$ can be found in [8]. All Riemannian k-symmetric spaces are Riemannian homogeneous so, by analogy with symmetric spaces, those which are naturally reductive should be of particular interest. As we show, there is a simple necessary and sufficient condition for this property to hold.

Riemannian locally k-symmetric spaces can be defined by means of particular local isometries or, more simply, by tensor conditions. When $k = 2$ these

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reduce to the single condition $\nabla R = 0$. For many purposes it is sufficient to consider local properties, since these extend globally when the further conditions of completeness and simple connectivity are imposed. Moreover, the property of being naturally reductive extends readily to Riemannian locally k-symmetric spaces.

On any Riemannian locally k-symmetric space (M, g) there exists a canonical F-structure which becomes an almost complex structure when k is odd [11]. In particular, when $k = 3$ and (M, g) is naturally reductive then F is a nearly Kahler structure. This raises the converse question of how to characterise (M, g) as a naturally reductive locally 3-symmetric space by means of its nearly Kahler structure. As shown by Gray in [3] and [4], just one further condition is necessary and this is a third-order one involving ∇R only. Thus, surprisingly perhaps, no algebraic condition on R is required.

Our purpose here is to consider the analogous problem for $k = 4$. In this case F does not reduce to an almost complex structure on (M, g) and each tangent space has a direct sum decomposition into two subspaces determined by F. In Section 2 we outline some basic properties of Riemannian k-symmetric spaces and, in particular, those which are naturally reductive. We then consider their local analogues which are our main concern. In Section 3 we use the first-order properties of a naturally reductive locally 4-symmetric space to define a $(4, F)$-manifold. It follows that a locally 4-symmetric space is naturally reductive if and only if it is a $(4, F)$-manifold. Then in Section 4 we consider higher-order conditions and obtain a characterisation of naturally reductive locally 4-symmetric spaces as a special class of $(4, F)$-manifolds. As before, these extra conditions are just on ∇R. The main results are given in Theorems 4.2, 4.3, 4.4 and 4.5.

One might conjecture that a 'useful' definition of a (k, F)-manifold would result via a study of naturally reductive locally k-symmetric spaces. From such a viewpoint nearly Kahler manifolds would be classified as $(3, F)$-manifolds. This more general question will be considered in a forthcoming paper.

For notational purposes we usually follow [6]. Any Riemannian manifold (M, g) is assumed to be connected, smooth and finite dimensional. In general, we write $\mathcal{T}^p_q$ for the algebra of smooth tensor fields with contravariant and covariant orders p and q respectively; in particular, we write $\mathcal{T}^p_0 = \mathcal{T}^p$ and $\mathcal{T}^0_p = \mathcal{T}_p$. Tensor fields $A, B \in \mathcal{T}^1_1$ will often be considered as linear endomorphisms and then composed in the usual way to give $AB \in \mathcal{T}^1_1$.

Finally, we remark that the results and proofs in this paper remain valid for pseudo-Riemannian manifolds.

2. Preliminaries

We recall here some basic properties of Riemannian k-symmetric spaces. Let (M, g) be a (smooth, connected) Riemannian manifold and let $s = \{s_x : x \in M\}$ be a family of isometries of (M, g) such that each $x \in M$ is an isolated fixed point of the corresponding map s_x. We call s_x a *Riemannian symmetry* at x and say (M, g) has a *regular* s-structure if

$$s_x \circ s_y = s_{s_x(y)} \circ s_x \qquad \text{for all } x, y \in M.$$

If, in addition, there exists a positive integer k such that, for each $x \in M$, k is the least positive integer for which s_x^k is the identity map on M, then we call (M, g) a (Riemannian) *k-symmetric space* and denote it by (M, g, s).

Any tensor field on (M, g, s) is called *s-invariant* if it is invariant under each symmetry s_x, $x \in M$. In particular, the tensor field $S \in \mathscr{T}_1^1$ defined by

$$(2.1) \qquad S_x X = s_{x_*} X \qquad \text{for each } x \in M \text{ and } X \in M_x$$

is smooth and s-invariant. We call S the *symmetry tensor field* on M and note that $I - S$ is non-singular at each point of M. Furthermore, we say a tensor field $T \in \mathscr{T}_q^p$ is S-invariant if, for all $w_1, \dots, w_p \in \mathscr{T}_1$ and $X_1, \dots, X_q \in \mathscr{T}^1$,

$$T(w_1 S, \dots, w_p S, X_1 \dots, X_q) = T(w_1, \dots, w_p, SX_1, \dots, SX_q)$$

where $(wS)X = w(SX)$ for $w \in \mathscr{T}_1$ and $X \in \mathscr{T}^1$. Thus it can be seen that the tensor fields $g, R, \nabla R, \nabla S$ and $\nabla^2 S$ are S-invariant where ∇ is the Riemannian connection on (M, g) and R the curvature tensor field.

Because of the regular s-structure on (M, g, s) we may consider it as a *reductive* homogeneous space [12] with respect to a group of isometries preserving the tensor field S. We write the corresponding *canonical connection* [2] as $\tilde{\nabla}$. Then $\tilde{\nabla}$ is invariant with respect to each symmetry s_x and $\tilde{\nabla} S = 0$. In fact, it is easily seen that these two properties characterise the canonical connection. Next, define $D \in \mathscr{T}_2^1$ by

$$(2.2) \qquad D_X Y = D(X, Y) = \nabla_X Y - \tilde{\nabla}_X Y \qquad \text{for all } X, Y \in \mathscr{T}^1.$$

Then D is invariant with respect to each s_x and, in particular, D is S-invariant. It follows that D is given by

$$(2.3) \qquad D_X Y = (\nabla_{(I-S)^{-1}X} S) S^{-1} Y \qquad \text{for all } X, Y \in \mathscr{T}^1.$$

Moreover, since $\tilde{\nabla} g = 0$ we see from (2.2) that each D_X, considered as a derivation, satisfies

$$D_X g = 0. \tag{2.4}$$

Now the homogeneous k-symmetric space (M, g, s) is *naturally reductive* with ∇ as the *natural* (torsion free) connection if and only if ∇ and $\tilde{\nabla}$ have the same geodesics [6]. From (2.2) this is equivalent to the condition

$$D_X X = 0 \qquad \text{for all } X \in \mathcal{T}^1. \tag{2.5}$$

Next, we consider local analogues. Thus, let (M, g) be a Riemannian manifold with a tensor field $S \in \mathcal{T}^1_1$ such that $I - S$ is non-singular and g is S-invariant. As before, we call S a *symmetry tensor field* and say S has *order k* if $S^k = I$ for some least positive integer k. Moreover, we say S is *regular* when the tensor fields ∇S and $\nabla^2 S$ are S-invariant. Suppose we are given S of order k on (M, g). Then for each $x \in M$ a *local symmetry* s_x of order k is defined on a sufficiently small neighbourhood of x by

$$s_x = \exp_x \circ S_x \circ \exp_x^{-1}.$$

Clearly, S_x and s_x are related as in (2.1). If there exists a family $\{s_x : x \in M\}$ of such maps for which each s_x is a local isometry preserving S, then we say (M, g) together with S is a (Riemannian) *locally k-symmetric space* and denote it by (M, g, S). Clearly k-symmetric spaces are locally k-symmetric. We recall the following results from [2].

THEOREM 2.1. *A Riemannian manifold (M, g) with symmetry tensor field S of order k is locally k-symmetric if and only if S is regular and the tensor fields R and ∇R are S-invariant. Moreover, if (M, g) is locally k-symmetric, complete and simply connected then it is k-symmetric.*

Suppose S is a regular symmetry tensor field on (M, g) and define D by (2.3). Then D and ∇D are S-invariant and we have

$$D_X g = 0 \qquad \text{for each } X \in \mathcal{T}^1. \tag{2.6}$$

Hence, from (2.2) we obtain a connection $\tilde{\nabla}$ which again we call the *canonical connection*. This can be shown to satisfy

$$\tilde{\nabla} g = \tilde{\nabla} S = \tilde{\nabla} D = 0. \tag{2.7}$$

In particular, we say a locally k-symmetric space (M, g, S) is *naturally reductive* if D satisfies (2.5). Henceforth, our purpose will be to consider such spaces when $k = 4$.

3. First-order conditions

Let (M, g, S) be a locally 4-symmetric space. Now $I - S$ is non-singular so we have

$$S^3 + S^2 + S + I = (S + I)(S^2 + I) = 0. \tag{3.1}$$

Define tensor fields F and P of type (1,1) by

$$F = \tfrac{1}{2}(S + I)^2 \tag{3.2}$$

and

$$P = \tfrac{1}{2}(S^2 + I) = F - S. \tag{3.3}$$

We call F the *canonical F*-structure on (M, g, S) (cf. [1] and [15]) noting that it satisfies

$$F^3 + F = 0 \tag{3.4}$$

and

$$g(FX, X) = 0 \qquad \text{for all } X \in \mathscr{T}^1. \tag{3.5}$$

Moreover,

$$P - F^2 = I, \qquad PF = FP = 0,$$

$$FS = SF = F^2, \qquad PS = SP = -P$$

and

$$g(FX, PX) = 0. \tag{3.6}$$

LEMMA 3.1. *Suppose (M, g, S) is a naturally reductive locally 4-symmetric space. Then the tensor fields F and P defined by* (3.2) *and* (3.3) *satisfy*

$$P(\nabla_X P)X = P(\nabla_X F)X = 0. \tag{3.7}$$

$$F(\nabla_{FX} F)FY = 0 \tag{3.8}$$

for all $X, Y \in \mathscr{T}^1$.

PROOF. We first consider corresponding properties for the derivation D_X, where we use (2.5) and the S-invariance of D. We have for all $X, Y \in \mathscr{T}^1$

$$PD_{FX}(PY) = PS^2 D_{FX}(PY) = -PD_{FX}(PY)$$

so

$$PD_{FX}(PY) = 0. \tag{3.9}$$

Then

$$P(D_X P)X = PD_X(PX) = PD_{(P-F^2)X}(PX) = 0. \tag{3.10}$$

Next, we note that

$$SD_{FX}(F^2X) = -D_{F^3X}(FX) = D_{FX}(F^2X),$$

hence

$$D_{FX}(F^2X) = 0. \tag{3.11}$$

Then

$$PD_X(FX) = PS^2D_X(FX) = -PD_{(2F^2+I)X}(FX)$$
$$= -PD_X(FX)$$

which implies

$$P(D_X F)X = 0. \tag{3.12}$$

Finally,

$$FD_{FX}(F^2Y) = -FS^2D_{FX}(F^2Y) = -FD_{FX}(F^2Y)$$

so

$$F(D_{FX}F)FY = 0. \tag{3.13}$$

Since $\bar{\nabla}F = \bar{\nabla}P = 0$, the lemma follows immediately from (2.2), (3.10), (3.12) and (3.13).

We now introduce the following definition.

DEFINITION. A $(4, F)$-manifold is a Riemannian manifold (M, g) together with a non-zero tensor field $F \in \mathcal{T}^1_1$, for which $P = I + F^2$ is non-zero and (3.4), (3.5), (3.7) and (3.8) are satisfied.

Let (M, g) be a (4, F)-manifold with Riemannian connection ∇, and define $S = F - P$ where $P = F^2 + I$. Then, as a consequence of (3.4), S has order 4 and $I - S$ is non-singular. Also, it follows using (3.5) that g is S-invariant and each tangent space M_x has an orthogonal direct sum decomposition induced by the projection maps $-F^2$ and P. Thus S is a symmetry tensor field of order 4 which we regard as the canonical symmetry tensor field on (M, g). We note that F and P are S-invariant since $F = \frac{1}{2}(S + I)^2$.

PROPOSITION 3.2. *On any $(4, F)$-manifold the tensor field ∇S is S-invariant.*

PROOF. Since $S = F - F^2 - I$ it is sufficient to show that ∇F is S-invariant. We need to consider several cases arising from the above orthogonal decomposition. For these, we use (3.7), (3.8) and the algebraic relations between F, P and S without further reference. Thus, the following properties hold for all $X, Y, Z \in \mathscr{T}^1$.

(i)

$$\begin{aligned} P(\nabla_{SX}F)SY &= P(\nabla_{SX}F^2)Y \\ &= -P(\nabla_Y F^2)SX \\ &= P(\nabla_Y F)X \\ &= -P(\nabla_X F)Y \\ &= SP(\nabla_X F)Y. \end{aligned} \tag{3.14}$$

Next we have

(ii)

$$\begin{aligned} g((\nabla_{SX}F)SPY, FZ) &= g(F(\nabla_{SX}P)Y, FZ) \\ &= -g((\nabla_{SX}P)Y, F^2Z) \\ &= g(PY, (\nabla_{SX}F^2)Z) \\ &= -g(PY, (\nabla_Z F^2)SX) \\ &= g(PY, (\nabla_Z F)X) \\ &= -g(PY, (\nabla_X F)Z) \\ &= g((\nabla_X P)Y, FZ) \\ &= g(F(\nabla_X F)PY, FZ) \\ &= g(S(\nabla_X F)PY, FZ). \end{aligned}$$

Thus

$$F(\nabla_{SX}F)SPY = SF(\nabla_X F)PY. \tag{3.15}$$

Finally,

(iii)

$$\begin{aligned} SF(\nabla_{PX}F)FY &= F^2(\nabla_{PX}F)FY \\ &= F(\nabla_{PX}F^2)FY - F(\nabla_{PX}F)F^2Y \\ &= F(\nabla_{PX}P)FY + F(\nabla_{SPX}F)SFY \\ &= F(\nabla_{SPX}F)SFY. \end{aligned} \tag{3.16}$$

By writing

$$(\nabla_X F)Y = P(\nabla_X F)Y - F^2(\nabla_X F)PY + F^2(\nabla_{PX} F)F^2 Y - F^2(\nabla_{F^2X} F)F^2 Y,$$

it follows from (3.8), (3.14), (3.15) and (3.16) that ∇F, hence ∇S, is S-invariant as required.

Next we define D on a $(4, F)$-manifold by (2.3). Then, as an easy consequence of (3.7), (3.8) and the S-invariance of ∇S, we have

COROLLARY 3.3. *On any $(4, F)$-manifold, D is S-invariant and $D_X X = 0$ for all $X \in \mathcal{T}^1$.*

From this result and Lemma 3.1 we obtain

PROPOSITION 3.4. *Let (M, g, S) be a locally 4-symmetric space. Then the following conditions are equivalent:*

(i) *(M, g, S) is naturally reductive,*

(ii) *(M, g, S) is a $(4, F)$-manifold with respect to its canonical F-structure.*

4. Higher-order conditions

In this section we adopt the following notation. We define $\nabla^2 S$ by

$$(\nabla^2 S)(X, Y, Z) = (\nabla^2_{XY} S)Z = \nabla_X(\nabla_Y S)Z - (\nabla_{\nabla_X Y} S)Z - (\nabla_Y S)\nabla_X Z$$

and define the covariant form of the curvature tensor field R by

$$R(X_1, X_2, X_3, X_4) = g(R(X_3, X_4)X_2, X_1).$$

Also, we extend the definition of S-invariant tensor fields to include tensor polynomials with entries in $\mathcal{T}^1$. Thus let $T \in \mathcal{T}_r$ and choose a partition of $\{1, \dots, r\}$ into disjoint subsets $B_1, \dots, B_l$. Define a *tensor polynomial* $p(U_1, \dots, U_l)$ on the l-fold product $\mathcal{T}^1 \times \cdots \times \mathcal{T}^1$ by $p(U_1, \dots, U_l) = T(X_1, \dots, X_r)$ where, for $i = 1, \dots, r$, $X_i = U_\alpha$ if $i \in B_\alpha$. Then we say $p(U_1, \dots, U_l)$ is *S-invariant* if $p(SU_1, \dots, SU_l) = p(U_1, \dots, U_l)$ for all $U_1, \dots, U_l \in \mathcal{T}^1$. Finally, for convenience of notation, we write $S^3 - I = A$.

We now consider S-invariant tensor polynomials arising from R.

LEMMA 4.1. *On any $(4, F)$-manifold (M, g) the curvature tensor field satisfies*

(i) *$R((P + I)X, FY, X, Y) + R(FX, (P + I)Y, X, Y)$ is S-invariant, and*

(ii) *$R(PX, FY, FY, F^2Y) = 0$.*

PROOF. (i) First note that, since g is S-invariant,

$$g((\nabla_X S)Z, SZ) = 0 \tag{4.1}$$

so

$$g((\nabla^2_{YX} S)Z, SZ) + g((\nabla_X S)Z, (\nabla_Y S)Z) = 0. \tag{4.2}$$

Also, from (2.3) and Corollary 3.3,

$$(\nabla_X S)AX = 0 \tag{4.3}$$

so

$$(\nabla^2_{YX} S)AX + (\nabla_X S)(\nabla_Y A)X = 0. \tag{4.4}$$

Then from the relations $SA = P + I - F$ and $A = -(P + I + F)$ we have

$$\begin{aligned} &2R((P+I)X, FY, X, Y) + 2R(FX, (P+I)Y, X, Y) \\ &= R(SAX, SAY, Y, X) - R(AX, AY, Y, X) \\ &= g((\nabla^2_{YX} S)AY - (\nabla^2_{XY} S)AY, SAX) \end{aligned}$$

which is S-invariant because of (4.2), (4.4) and the S-invariance of ∇S.

(ii) From (3.7) we have

$$g((\nabla_{FY} S)FY, PX) = g((\nabla_{F^2Y} S)FY, PX) = 0$$

and it follows, as above, that $g((\nabla^2_{F^2YFY} S)FY, PX)$ and $g((\nabla^2_{FYF^2Y} S)FY, PX)$ are S-invariant. Hence $R(PX, F^2Y, F^2Y, FY) + R(PX, FY, F^2Y, FY)$ is S-invariant, which implies $R(PX, FY, FY, F^2Y) = 0$.

THEOREM 4.2. *The curvature tensor field on any* (4, F)*-manifold* (M, g) *is* S*-invariant.*

PROOF. Let X, Y, Z, W denote arbitrary vector fields. It is clearly sufficient to prove S-invariance for the four cases

(i) $R(PX, PY, PZ, FW)$, (ii) $R(PX, FY, PZ, FW)$,
(iii) $R(PX, FY, FZ, FW)$, (iv) $R(FX, FY, FZ, FW)$,

which we now do.

(i) By writing X as PX and Y as $PY + FY$ in (i) of Lemma 4.1 we have $R(PX, PY, PX, FY) = 0$. Hence $R(PX, PY, PZ, FW) = 0$.

(ii) Write X as $PX + F^2X$ and Y as $PY + FY$ in (i) of Lemma 4.1. Then

$$\begin{aligned}(4.5)\qquad & 2R(PX, F^2Y, F^2X, PY) + R(PX, PY, F^2X, F^2Y) \\ & - 2R(PX, FY, FX, PY) - R(PX, PY, FX, FY) = 0\end{aligned}$$

since the left-hand side of (4.5) is S-invariant. In particular, by writing FY as FX in (4.5) and linearising we obtain

$$\begin{aligned}(4.6)\qquad & R(PX, F^2Y, F^2W, PZ) + R(PX, F^2W, F^2Y, PZ) \\ & - R(PX, FY, FW, PZ) - R(PX, FW, FY, PZ) = 0.\end{aligned}$$

Then from (4.5) and the first Bianchi identity,

$$\begin{aligned} & 3R(PX, F^2Y, F^2W, PZ) - 3R(PX, FY, FW, PZ) \\ & - R(PX, F^2W, F^2Y, PZ) + R(PX, FW, FY, PZ) = 0.\end{aligned}$$

This equation and (4.6) imply that $R(PX, FY, FW, PZ)$ is S-invariant which proves (ii).

(iii) By linearising (ii) of Lemma 4.1 we obtain

$$R(PX, FX, FY, F^2Y) + R(PX, FY, FX, F^2Y) + R(PX, FY, FY, F^2X) = 0.$$

Now apply the first Bianchi identity to get

$$\begin{aligned}(4.7)\qquad & 2R(PX, FY, FX, F^2Y) - R(PX, F^2Y, FX, FY) \\ & + R(PX, FY, FY, F^2X) = 0.\end{aligned}$$

Also, from (i) of Lemma 4.1,

$$2R(PX, F^2Y, FX, FY) + R(PX, FY, FX, F^2Y) + R(PX, FY, F^2X, FY)$$

is S-invariant. This, together with (4.7), shows that

$$3R(PX, FY, FX, F^2Y) + R(PX, F^2Y, FX, FY)$$

is S-invariant. The same is true with FY in place of Y, so it follows that $R(PX, FY, FX, F^2Y)$ is S-invariant and hence zero. Then from (4.7), $R(PX, FY, FY, F^2X) = 0$ and (iii) is an easy consequence.

(iv) Since $F(\nabla_{FW}S)FY = 0$ we see that $(\nabla^2_{FZFW}S)FY$ is S-invariant. Consequently,

$$\begin{aligned} & R(F^2X, F^2Y, FZ, FW) - R(FX, FY, FZ, FW) \\ & = g((\nabla^2_{FZFW}S)FY - (\nabla^2_{FWFZ}S)FY, F^2X)\end{aligned}$$

is S-invariant. Then

$$R(F^2X, F^2Y, F^2Z, F^2W) - R(FX, FY, FZ, FW)$$

is S-invariant and hence zero, which proves (iv) and completes the proof of the theorem.

THEOREM 4.3. *The symmetry tensor field S on any $(4, F)$-manifold is regular.*

PROOF. Because of Proposition 3.2, it remains only to prove that $\nabla^2 S$ is S-invariant. Now since

$$(\nabla^2_{XY} S)Z - (\nabla^2_{YX} S)Z = R(X, Y)SZ - SR(X, Y)Z$$

and R is S-invariant, it suffices to prove the S-invariance of $\nabla^2_{XX} S$. But from (4.4) we have

$$(\nabla^2_{XX} S)AY = -((\nabla^2_{XY} S)AX - (\nabla^2_{YX} S)AX) - (\nabla_X S)(\nabla_X A)Y$$
$$- (\nabla_Y S)(\nabla_X A)X - (\nabla_X S)(\nabla_Y A)X,$$

and the right-hand side of this equation is clearly S-invariant due to Proposition 3.2 and Theorem 4.2. This completes the proof.

We now obtain third-order conditions which are necessary and sufficient for a Riemannian $(4, F)$-manifold to be locally 4-symmetric.

THEOREM 4.4. *Let (M, g) be any $(4, F)$-manifold and suppose, for all $X, Y, Z, U, V \in \mathscr{T}^1$,*

(i) $(\nabla_{PX} R)(PY, PZ, PU, PV) = 0$,

(ii) $(\nabla_{FX} R)(FY, FZ, FU, FV) = 0$.

Then (M, g) is a naturally reductive locally 4-symmetric space (M, g, S) for which F is the canonical F-structure. Conversely with respect to its canonical F-structure, any naturally reductive locally 4-symmetric space is a $(4, F)$-manifold for which (i) *and* (ii) *are satisfied.*

PROOF. Suppose (M, g) is a $(4, F)$-manifold on which ∇R satisfies (i) and (ii). We prove that ∇R is S-invariant. Because of the algebraic identities satisfied by ∇R, we need consider only five cases. These are

(iii) $(\nabla_X R)(FY, FZ, FU, PV)$, (iv) $(\nabla_X R)(FY, PZ, PU, PU)$,

(v) $(\nabla_{PX} R)(FY, FZ, FU, FV)$, (vi) $(\nabla_{FX} R)(PY, PZ, PU, PV)$,

(vii) $(\nabla_X R)(FY, PZ, FU, PV)$,

for all X, Y, Z, U, V. For (iii) we first note that, from the S-invariance of R,

$R(FY, FZ, FU, PV) = 0$, as can be seen by acting with S^2. Then (iii) follows using the S-invariance of R and ∇S. Similarly, we obtain (iv) by noting that $R(FY, PZ, PU, PV) = 0$. Then (v) and (vi) are immediate consequences of (iii) and (iv) respectively and the second Bianchi identity. Finally, to prove (vii) we first note that $g((\nabla_{PV}S)FY, PZ) = 0$, from which

$$\begin{aligned} &g((\nabla^2_{UPV}S)FY + (\nabla_{\nabla_U P)V}S)FY + (\nabla_{PV}S)(\nabla_U F)Y, PZ) \\ &\quad + g((\nabla_{PV}S)FY, (\nabla_U P)Z) = 0. \end{aligned} \tag{4.8}$$

Also (3.7) implies $g((\nabla_{FY}S)FY, PZ) = 0$, hence

$$\begin{aligned} &g((\nabla^2_{UFY}S)FY + (\nabla_{\nabla_U F)Y}S)FY + (\nabla_{FY}S)(\nabla_U F)Y, PZ) \\ &\quad + g((\nabla_{FY}S)FY, (\nabla_U P)Z) = 0. \end{aligned} \tag{4.9}$$

Then from (4.8) and (4.9)

$$\begin{aligned} &R(PZ, F^2Y, FY, PV) + R(PZ, FY, FY, PV) \\ &= g((\nabla^2_{FYPV}S)FY - (\nabla^2_{PVFY}S)FY, PZ) \\ &= -g((\nabla_{(V_{FY}P)V}S)FY + (\nabla_{PV}S)(\nabla_{FY}F)Y, PZ) \\ &\quad - g((\nabla_{PV}S)FY, (\nabla_{FY}P)Z) \\ &\quad + g((\nabla_{(\nabla_{PV}F)Y})FY + (\nabla_{FY}S)(\nabla_{PV}F)Y, PZ) + g((\nabla_{FY}S)FY, (\nabla_{PV}P)Z). \end{aligned} \tag{4.10}$$

Similarly, since $g((\nabla_{FY}S)F^2Y, PZ) = 0$,

$$\begin{aligned} &g((\nabla^2_{UFY}S)F^2Y + (\nabla_{(\nabla_U F)Y}S)F^2Y + (\nabla_{FY}S)(\nabla_U F^2)Y, PZ) \\ &\quad + g((\nabla_{FY}S)F^2Y, (\nabla_U P)Z) = 0. \end{aligned} \tag{4.11}$$

Hence, from (4.8) and (4.11)

$$\begin{aligned} &R(PZ, FY, FY, PV) - R(PZ, F^2Y, FY, PV) \\ &= -g((\nabla^2_{FYPV}S)F^2Y - (\nabla^2_{PVFY}S)F^2Y, PZ) \\ &= g((\nabla_{(\nabla_{FY}P)V}S)F^2Y + (\nabla_{PV}S)(\nabla_{FY}F)FY, PZ) \\ &\quad + g((\nabla_{PV}S)F^2Y, (\nabla_{FY}P)Z) - g((\nabla_{(\nabla_{PV}F)Y}S)F^2Y \\ &\quad + (\nabla_{FY}S)(\nabla_{PV}F^2)Y, PZ) \\ &\quad - g((\nabla_{FY}S)F^2Y, (\nabla_{PV}P)Z). \end{aligned} \tag{4.12}$$

Now, from (4.10) and (4.12) we see that $R(PZ, FY, FY, PV)$ can be written explicitly in terms of ∇S alone, thus no second-order terms involving $\nabla^2 S$ are required. It follows that $(\nabla_X R)(PZ, FY, FY, PV)$ is S-invariant for all $X, Z, Y, V \in \mathscr{T}^1$. Next, we prove $(\nabla_X R)(FY, FZ, PZ, PV)$ is S-invariant. Thus

$$(\nabla_{FX} R)(FY, FZ, PZ, PV)$$

$$= -(\nabla_{PV} R)(FY, FZ, FX, PZ) - (\nabla_{PZ} R)(FY, FZ, PV, FX)$$

which is S-invariant by (iii). Similarly, it follows using (iv) that $(\nabla_{PX} R)(FY, FZ, PZ, PV)$ is S-invariant. Hence $(\nabla_X R)(FY, FZ, PZ, PV)$ is S-invariant. From this property and the S-invariance of $(\nabla_X R)(PZ, FY, FY, PV)$, as proved above, we readily obtain (vii). Thus we see from Corollary 3.3 and Theorems 2.1, 4.2 and 4.3 that (M, g) is a naturally reductive locally 4-symmetric space for which F is canonical.

The converse follows from Proposition 3.4 and by noting that (i) and (ii) are an immediate consequence of ∇R being S-invariant.

We remark that in the definitions of a locally 4-symmetric space and a $(4, F)$-manifold the cases $F = 0$, $P = 0$ are excluded. This avoids the occurrence of locally symmetric spaces (when $F = 0$) and Hermitian locally symmetric spaces (when $P = 0$) as shown by (3.8) and (i), (ii) of Theorem 4.4.

Finally, from Theorem 2.1 and Theorem 4.4 we have the following immediate global result.

THEOREM 4.5. *Any complete, simply connected* $(4, F)$*-manifold* (M, g) *for which* (i) *and* (ii) *of Theorem* 4.4 *are satisfied is a naturally reductive* 4*-symmetric space* (M, g, s) *for which F is the canonical F-structure.*

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